

EDGE DISTRIBUTION AND DENSITY IN THE CHARACTERISTIC SEQUENCE

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ABSTRACT. The characteristic sequence of hypergraphs $\langle P_n : n < \omega \rangle$ associated to a formula $\varphi(x; y)$, introduced in [6], is defined by $P_n(y_1, \dots, y_n) = (\exists x) \bigwedge_{i \leq n} \varphi(x; y_i)$. This paper continues the study of characteristic sequences, showing that graph-theoretic techniques, notably Szemerédi’s celebrated regularity lemma, can be naturally applied to the study of model-theoretic complexity via the characteristic sequence. Specifically, we relate classification-theoretic properties of φ and of the P_n (considered as formulas) to density between components in Szemerédi-regular decompositions of graphs in the characteristic sequence. In addition, we use Szemerédi regularity to calibrate model-theoretic notions of independence by describing the depth of independence of a constellation of sets and showing that certain failures of depth imply Shelah’s strong order property SOP_3 ; this sheds light on the interplay of independence and order in unstable theories.

1. INTRODUCTION

The characteristic sequence $\langle P_n : n < \omega \rangle$ is a tool for studying the combinatorial complexity of a given formula φ , Definition 2.2 below. It follows from [5], [6] that the Keisler order [3] localizes to the study of φ -types and specifically of characteristic sequences. However, this article will not focus on ultrapowers.

The analysis of [6] established that characteristic sequences are essentially trivial when the ambient theory T is *NIP*, Theorem 2.8 below. In this article, we turn to the study of characteristic sequences in the presence of the independence property. The framework of characteristic sequences allows us to bring a deep collection of graph-theoretic structure theorems to bear on our investigations. Notably, the classic model-theoretic move of polarizing complex structure into rigid and random components (e.g. Shelah’s isolation of the independence property and the strict order property in unstable theories) is accomplished here by the application of Szemerédi’s Regularity Lemma, §4 Theorem B below. Because the Regularity Lemma describes a possible decomposition of *any* sufficiently large graph, it can be applied here to understand how arbitrarily large subsets of P_1 generically interrelate.

In Sections 3-5, we investigate how classic properties of T affect the density δ attained between arbitrarily large ϵ -regular subsets $A, B \subset P_1$ (after localization) in the sense of Szemerédi regularity, where the edge relation is given by P_2 . The picture we obtain is as follows. When φ is stable, by Theorem 2.8, the density (after localization) is always 1. When φ is simple unstable, after localization, there will be an infinite number of missing edges but we can say something strong about their distribution: (*) the density between arbitrarily large ϵ -regular pairs must tend towards 0 or 1 as the graphs grow (indeed, here simplicity is sufficient but not necessary). In the simple unstable case, a finer function counting the number of edges omitted over finite subgraphs of size n is meaningful, and we give a preliminary description of its possible values in Proposition 3.9. In Section 5, we

use model theory to relate the property $(*)$ of having arbitrarily large ϵ -regular subsets of P_1 with edge density bounded away from 0 and 1 to the phenomenon of instability in the characteristic sequence, which is strictly more complex than failure of simplicity. In Section 6 we refine this phenomenon by defining and investigating the compatible and empty order properties. On the level of theories, the compatible order property characterizes the model-theoretic rigidity property SOP_3 , which is known to imply maximality in the Keisler order by [8].

In the other direction, in Section 7 we use Szemerédi regularity to bring to light a subtle model-theoretic failure of randomness, by considering the “depth of independence” of a constellation of infinite sets. In the language of Definition 7.2, we show that theories which are I_n^{n+1} but not I_{n+1}^{n+1} for some $n > 2$, are SOP_3 . This is a result about the fine structure of the classic SOP/IP distinction, illustrating the tradeoff between a weaker notion of strict order (SOP_3) and a stronger notion of independence (I_{n+1}^{n+1}) in unstable theories.

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2. PRELIMINARIES

The following conventions will be in place throughout the article.

Convention 2.1. (Conventions)

- (1) *If a variable or a tuple is written x or a rather than $\overline{x}, \overline{a}$, this does not necessarily imply that $\ell(x), \ell(a) = 1$.*
- (2) *Unless otherwise stated, T is a complete theory in the language \mathcal{L} .*
- (3) *A set is k -consistent if every k -element subset is consistent, and it is k -inconsistent if every k -element subset is inconsistent.*
- (4) $\varphi_\ell(x; y_1, \dots, y_\ell) := \bigwedge_{i \leq \ell} \varphi(x; y_i)$
- (5) $S_{\aleph_0}(\omega)$ is the set of all finite subsets of ω .
- (6) ϵ, δ are real numbers, with $0 < \epsilon < 1$ and $0 \leq \delta \leq 1$.
- (7) *Let G be a symmetric binary graph. We present graphs model-theoretically, i.e. as sets of vertices on which certain edge relations hold. Throughout this article $R(x, y)$ is a binary edge relation, which will sometimes (we will clearly say when) be interpreted as P_2 .*
- (8) *A graph is a simple graph: no loops and no multiple edges. Definition 2.2 below implies that $\forall x(P_1(x) \rightarrow P_2(x, x))$, but we will, by convention, not count loops when taking P_2 as R .*
- (9) *Given a graph G , with symmetric binary edge relation $R(x, y)$:*
 - $|G|$ is the size of G , i.e. the number of vertices.
 - $e(G)$ is the number of edges of G .
 - $\hat{e}(G)$ is the number of edges omitted in G .
 - An empty graph is a graph with no edges.
 - A complete graph is a graph with all edges, i.e. in which $x, y \in G, x \neq y \implies R(x, y)$.
 - The degree of a vertex is the number of edges which contain it.

- The dual graph G' has the same vertices and inverted edges, i.e. for $x \neq y$, $G' \models R(x, y) \iff G \models \neg R(x, y)$.
- (10) Write (X, Y) to indicate a bipartite graph. Then:
- $e(X, Y)$ is the number of edges between elements $x \in X$ and $y \in Y$. Note that if $G = A \cup B$ then possibly $e(G) \neq e(A, B)$, as the latter counts only edges between A and B .
 - $\hat{e}(X, Y)$ is the number of edges omitted between elements $x \in X$ and $y \in Y$.
 - The density of a finite bipartite graph (X, Y) is $\delta(X, Y) := e(X, Y)/|X||Y|$ when $|X|, |Y| \neq 0$, and 0 otherwise.
 - An empty pair is a pair of vertices x, y with $\neg R(x, y)$.
 - An infinite empty pair is (X, Y) such that $|X| = |Y| \geq \aleph_0$ and for all $x \in X, y \in Y$, we have $\neg R(x, y)$.
 - A complete bipartite graph is (X, Y) such that for all $x \in X, y \in Y$, $R(x, y)$.
 - The dual $(X, Y)'$ of a bipartite graph inverts precisely the edges between the components X and Y .

We will make extensive use of the important classification-theoretic dividing lines of stability, simplicity, the independence property, and the strict order property; see, for instance, [7], Chapter II, sections 2-4 and [8]. A theory or a formula is NIP, also called dependent, if it does not have the independence property; see, for instance, [11].

We now turn to definitions. The characteristic sequence of hypergraphs was introduced in [6] as a tool for studying the complexity of a given formula φ . Let us set the stage by briefly reviewing some of the results obtained there.

Definition 2.2. (Characteristic sequences) *Let T be a first-order theory and φ a formula of the language of T .*

- For $n < \omega$, $P_n(z_1, \dots, z_n) := \exists x \bigwedge_{i \leq n} \varphi(x; z_i)$.
- The characteristic sequence of φ in T is $\langle P_n : n < \omega \rangle$.
- Write $(T, \varphi) \mapsto \langle P_n \rangle$ for this association.
- We assume that $T \vdash \forall y \exists z \forall x (\varphi(x; z) \leftrightarrow \neg \varphi(x; y))$. If this does not already hold for some given φ , replace φ with $\theta(x; y, z) = \varphi(x; y) \wedge \neg \varphi(x; z)$.

Convention 2.3. As the characteristic sequence is definable in T , its first-order properties depend only on the theory and not on the model of T chosen. Throughout this paper, we will be interested in whether certain, possibly infinite, configurations appear as subgraphs of the P_n . By this we will always mean *whether or not it is consistent with T that such a configuration exists when P_n is interpreted in some sufficiently saturated model*. Thus, without loss of generality the formulas P_n will often be identified with their interpretations in some monster model.

Characteristic sequences give a natural context for studying the complexity of φ -types, which correspond in this case to complete graphs.

Definition 2.4. *Fix T, φ , $M \models T$ and $(T, \varphi) \mapsto \langle P_n \rangle$.*

- (1) A positive base set is a set $A \subset P_1$ such that $A^n \subset P_n$ for all $n < \omega$.
- (2) The sequence $\langle P_n \rangle$ has support k if: $P_n(y_1, \dots, y_n)$ iff P_k holds on every k -element subset of $\{y_1, \dots, y_n\}$. The sequence has finite support if it has support k for some $k < \omega$.

- (3) The elements $a_1, \dots, a_k \in P_1$ are a k -point extension of the P_∞ -complete graph A just in case Aa_1, \dots, a_k is also a P_∞ -complete graph.

Observation 2.5. Fix T, φ and $M \models T$ and suppose $(T, \varphi) \mapsto \langle P_n \rangle$.

- (1) The following are equivalent, for a set $A \subset M$:
 - (a) A is a positive base set.
 - (b) The set $\{\varphi(x; a) : a \in A\}$ is consistent.
- (2) The following are equivalent, for a set $A \subset P_1$:
 - (a) $A^n \cap P_n = \emptyset$ for some n .
 - (b) $\{\varphi(x; a) : a \in A\}$ is 1-consistent but n -inconsistent (Convention 2.1(2)).

Note that if A is infinite, compactness then implies some instance of φ divides.
- (3) The following are equivalent:
 - (a) $\langle P_n \rangle$ has finite support.
 - (b) φ does not have the finite cover property.

Localization is a definable restriction of the predicates P_n of a certain useful form which eliminates some of the combinatorial noise around a positive base set A under analysis. Definability ensures that Convention 2.3 applies when asking whether certain configurations are present in some localization.

Definition 2.6. (Localization, Definition 5.1 of [6]) Fix a characteristic sequence $(T, \varphi) \rightarrow \langle P_n \rangle$, and choose $B, A \subset M \models T$ with A a positive base set, possibly empty. A localization P_n^f of the predicate $P_n(y_1, \dots, y_n)$ around the positive base set A with parameters from B is given by a finite sequence of triples $f : m \rightarrow \omega \times \mathcal{P}_{\aleph_0}(y_1, \dots, y_n) \times \mathcal{P}_{\aleph_0}(B)$ where $m < \omega$ and:

- writing $f(i) = (r_i, \sigma_i, \beta_i)$ and \check{s} for the elements of the set s , we have:

$$P_n^f(y_1, \dots, y_n) := \bigwedge_{i \leq m} P_{r_i}(\check{\sigma}_i, \check{\beta}_i)$$

- for each $\ell < \omega$, T_1 implies that there exists a P_ℓ -complete graph C_ℓ such that P_n^f holds on all n -tuples from C_ℓ . If this last condition does not hold, P_n^f is a trivial localization. By localization we will always mean non-trivial localization.
- In any model of T_1 containing A and B , P_n^f holds on all n -tuples from A .

For the purposes of this article, we will indicate where localization is useful without, generally, specifying the parameters or the form involved, writing simply “there exists a localization in which...” or “after localization...” for short. Because this may always be taken to include a fixed positive base set, the essential complexity of the type under analysis is not lost. Localization reveals a gap in the classification-theoretic complexity of φ and of P_2 . §5 below will shed light on this result:

Conclusion 2.7. (Conclusion 5.10 of [6]) Suppose T is simple, $(T, \varphi) \mapsto \langle P_n \rangle$. Then for any $n < \omega$, and any partition of y_1, \dots, y_n into object and parameter variables, after localization the formulas $P_2(y_1, y_2), \dots, P_n(y_1, \dots, y_n)$ do not have the order property.

It turns out that when φ is NIP one can always localize (without losing sight of the positive base set A under analysis) so that any given finite initial segment of the characteristic sequence is a complete graph. In other words, the characteristic sequence is non-trivial in the presence of the independence property.

Theorem 2.8. (Theorem 6.17 of [6]) *Let φ be a formula of T and $\langle P_n \rangle$ its characteristic sequence.*

- (1) *If φ is NIP, then for each positive base set $A \subset P_1$ and for each $n < \omega$, there exists a localization $P_1^{f_n} \supset A$ of P_1 which is a P_n -complete graph, i.e. $\{y_1, \dots, y_n\} \subset P_1^{f_n} \rightarrow P_n(y_1, \dots, y_n)$.*
- (2) *If φ has IP, then for all $n < \omega$, P_1 contains a P_n -empty tuple.*

Furthermore, when φ is simple unstable we may assume that after localization, in any given finite initial segment of the characteristic sequence, there are uniform finite bounds on the size of empty graphs.

Theorem 2.9. (Theorem 6.24 of [6]) *Let φ be a formula of T and $\langle P_n \rangle$ its characteristic sequence.*

- (1) *If φ is simple, then for each P_∞ -graph $A \subset P_1$ and for each $n < \omega$, there exists a localization $P_1^{f_n} \supset A$ of P_1 in which there is a uniform finite bound on the size of a P_n -empty graph, i.e. there exists m_n such that $X \subset P_1$ and $X^n \cap P_n = \emptyset$ implies $|X| \leq m_n$.*
- (2) *If φ is not simple, then for all but finitely many $r < \omega$, P_1 contains an infinite $(r+1)$ -empty graph.*

The stage is now set as follows. The characteristic sequence of hypergraphs are a sequence of incidence relations defined on the parameter space of a formula φ . Positive base sets correspond naturally (though not necessarily uniquely) to base sets for φ -types. We turn to the study of the generic interrelationships between sets generally, and positive base sets particularly, in the parameter space of a given φ . Theorem 2.8 strongly focuses our attention on the “wild” case of theories with the independence property and Theorem 2.9 suggests simple unstable theories as a first object of study.

3. COUNTING FUNCTIONS ON SIMPLE φ

Throughout this section, we consider the binary edge relation P_2 from the characteristic sequence of φ . The notation and vocabulary follow Convention 2.1.

Observation 3.1. *Suppose φ is stable. Then after localization, for any two disjoint finite $X, Y \subset P_1$, $\delta(X, Y) = 1$. On the other hand, if φ is simple unstable then P_1 contains an empty pair.*

Proof. Theorem 2.8(1) says that when φ is stable, after localization P_1 is a complete graph, so a fortiori there are no edges omitted between disjoint components. The second clause is Theorem 2.8(2). \square

Definition 3.2. *Define $\alpha : \omega \rightarrow \omega$ to be*

$$\max \{ \hat{e}(X) : X \subset P_1, |X| = n \}$$

i.e. the largest number of P_2 -edges omitted over an n -size subset of P_1 .

Observation 3.3. *Suppose φ is simple, i.e., φ does not have the tree property. Then after localization $\alpha(n) < \frac{n(n-1)}{2}$.*

Proof. The maximum possible value $\frac{n(n-1)}{2}$ of any $\alpha(n)$ is attained on a P_2 -empty graph, on which $x \neq y \implies \neg P_2(x, y)$. Apply Theorem 2.9 which says that when φ does not have the tree property then we have, after localization, a uniform finite bound k on the size of a P_2 -empty graph $X \subset P_1$. So the function α is eventually strictly below the maximum. \square

Corollary 3.4. *The function $\alpha(n)$ is meaningful, i.e. after localization*

$$\frac{n(n-1)}{2} > \alpha(n) > 0$$

precisely when φ is simple unstable.

With some care we can easily restrict the range further. A famous theorem of Turán says:

Theorem A. (Turán, [4]:Theorem 2.2) *If G_n is a graph with n vertices and*

$$e(G) > \left(1 - \frac{1}{k-1}\right) \frac{n^2}{2}$$

then G_n contains a complete subgraph on k vertices.

Definition 3.5. $X = \langle a_i^t : t < 2, i < \omega \rangle \subset P_1$ *is an $(\omega, 2)$ -array if for all $n < \omega$,*

$$P_n(a_{i_1}^{t_1}, \dots, a_{i_n}^{t_n}) \iff (\forall j, \ell \leq n) (i_j = i_\ell \implies t_j = t_\ell)$$

Claim 3.6. (Claim 4.5 of [6]) *The following are equivalent, for a formula φ with characteristic sequence $\langle P_n \rangle$:*

- (1) φ *has the independence property.*
- (2) $\langle P_n \rangle$ *has an $(\omega, 2)$ -array.*

Observation 3.7. *Suppose that $\langle P_n \rangle$ has an $(\omega, 2)$ -array. Then $\alpha(n) \geq \lfloor \frac{n}{2} \rfloor$.*

Corollary 3.8. *When φ is simple unstable, then after localization*

$$\left(1 - \frac{1}{k-1}\right) \frac{n^2}{2} \geq \alpha(n) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

Proof. If φ is simple unstable, φ has the independence property and so P_1 contains an $(\omega, 2)$ -array; so the righthand side is Observation 3.7. For the lefthand side, let $k > 1$ be the uniform finite bound on the size of an empty graph from Theorem 2.9, and apply Turán's theorem to the dual graph. \square

At the end of Section 4 we will give a proof of the following:

Proposition 3.9. *When φ is simple unstable either*

$$\left(1 - \frac{1}{1-k}\right) \frac{n^2}{2} \geq \alpha(n) \geq \frac{n^2}{4} \quad \text{or} \quad \mathcal{O}(n^2) > \alpha(n) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

The proof will follow from Theorem 4.8 below, which will show more, namely that for φ simple unstable, either $\mathcal{O}(n^2) > \alpha(n)$ or there exists an infinite empty pair in P_1 .

Our strategy is going to be to show that in the absence of such an “empty pair” we can repeatedly partition sufficiently large graphs into many pieces of roughly equal size in such a way that, at each stage, the bulk of the omitted edges must occur inside the (eventually, much smaller) pieces. The main tool will be Theorem B below.

4. SZEMERÉDI REGULARITY

We begin with a review of Szemerédi's celebrated *regularity lemma*. Recall that ϵ, δ are real numbers, $0 < \epsilon < 1$ and $0 \leq \delta \leq 1$, following Convention 2.1.

Definition 4.1. [10], [4] *The finite bipartite graph (X, Y) is ϵ -regular if for every $X' \subset X$, $Y' \subset Y$ with $|X'| \geq \epsilon|X|$, $|Y'| \geq \epsilon|Y|$, we have: $|\delta(X, Y) - \delta(X', Y')| < \epsilon$.*

The regularity lemma says that sufficiently large graphs can always be partitioned into a fixed finite number of pieces X_i of approximately equal size so that almost all of the pairs (X_i, X_j) are ϵ -regular.

Theorem B. (Szemerédi's Regularity Lemma [4], [10]) *For every ϵ, m_0 there exist $N = N(\epsilon, m_0)$, $m = m(\epsilon, m_0)$ such that for any graph X , $N \leq |X| < \aleph_0$, for some $m_0 \leq k \leq m$ there exists a partition $X = X_1 \cup \dots \cup X_k$ satisfying:*

- $||X_i| - |X_j|| \leq 1$ for $i, j \leq k$
- All but at most ϵk^2 of the pairs (X_i, X_j) are ϵ -regular.

One important consequence is that we may, approximately, describe large graphs G as random graphs where the edge probability between x_i and x_j is the density $d_{i,j}$ between components X_i, X_j in some Szemerédi-regular decomposition. We include here two formulations of this idea from the literature, the first for intuition and the second for our applications.

Theorem C. (from Gowers [2]) *For every $\alpha > 0$ and every k there exists $\epsilon > 0$ with the following property. Let V_1, \dots, V_k be sets of vertices in a graph G , and suppose that for each pair (i, j) the pair (V_i, V_j) is ϵ -regular with density δ_{ij} . Let H be a graph with vertex set (x_1, \dots, x_k) and let $v_i \in V_i$ be chosen uniformly at random, the choices being independent. Then the probability that $v_i v_j$ is an edge of G iff $x_i x_j$ is an edge of H differs from $\prod_{x_i x_j \in H} \delta_{ij} \prod_{x_i x_j \notin H} (1 - \delta_{ij})$ by at most α .*

The formulation we will use, Theorem D, requires a preliminary definition.

Definition 4.2. [4] (The reduced graph)

- (1) Let $G = X_1, \dots, X_k$ be a partition of the vertex set of G into disjoint components. Given parameters ϵ, δ , define the reduced graph $R(G, \epsilon, \delta)$ to be the graph with vertices x_i ($1 \leq i \leq k$) and an edge between x_i, x_j just in case the pair (X_i, X_j) is ϵ -regular of density $\geq \delta$.
- (2) Write $R(t)$ for a full graph of height t whose reduced graph is R , i.e., $R(t)$ consists of k components X_1, \dots, X_k , each with t vertices, such that $e(X_i) = 0$, and $\delta(X_i, X_j) = 1$ iff there is an edge between x_i and x_j in R .

The following lemma (called the “Key Lemma” in [4]) says that sufficiently small subgraphs of the reduced graph must actually occur in the original graph G .

Theorem D. (Key Lemma, [4]:Theorem 2.1) *Given $\delta > \epsilon > 0$, a graph R , and a positive integer m , let G be any graph whose reduced graph is R , and let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$. Set $d = \delta - \epsilon$ and $\epsilon_0 = d^\Delta / (2 + \Delta)$. Then if $\epsilon \leq \epsilon_0$ and $t - 1 \leq \epsilon_0 m$, then $H \subset G$. Moreover the number of copies of H in G is at least $(\epsilon_0 m)^h$.*

Remark 4.3. In the statement of the Key Lemma, “ $H \subset G$ ” means that there is a bijection $f : H \rightarrow X \subset G$ such that $e(h_1, h_2)$ implies $e(f(h_1), f(h_2))$. With some slight modifications (recording whether a missing edge in the reduced graph means the density is near 0 or the pair is not regular; and using the dual graphs when necessary) we may assume “ $H \subset G$ ” has the usual meaning of isomorphic embedding, but this will not be an issue for the arguments in this section.

We now work towards a proof of Proposition 3.9.

Convention 4.4. (Interstitial edges, $b_{\epsilon, \ell}$, $N_{\epsilon, \ell}$, $E_{\epsilon, \ell}$)

- (1) Let G be a graph and let $G = X_1 \cup \dots \cup X_n$ be a decomposition into disjoint components, for instance as given by Theorem B. Call any edge between vertices $x \in X_i, z \in X_j, i \neq j$ an interstitial edge.
- (2) Let $b_{\epsilon, \ell}$ denote the upper bound on the necessary number of components, given by the regularity lemma as a function of ϵ, ℓ .
- (3) Write $(\epsilon, \ell)^*$ -decomposition to denote any Szemerédi-regular decomposition into k components, for any $\ell \leq k \leq b_{\epsilon, \ell}$.
- (4) Let $N_{\epsilon, \ell}$ denote the threshold size given by the regularity lemma as a function of ϵ, ℓ , such that any graph X with $|X| > N_{\epsilon, \ell}$ admits an $(\epsilon, \ell)^*$ -decomposition.
- (5) Let $E_{\epsilon, \ell} \subset \omega$ (E for “exactly”) be the (possibly empty) set of cardinalities n for which $|X| = n$ implies X admits a Szemerédi-regular decomposition into exactly ℓ components. See the next Remark.

Remark 4.5. On Definition 4.4(2)-(4): the Regularity Lemma, along with the pigeonhole principle, implies that for cofinally many ℓ , $E_{\epsilon, \ell}$ is infinite. Often, as Corollary 4.7(3) suggests, for the purposes of our asymptotic argument it is sufficient to know that the number of components fluctuates in a certain fixed range, as given by the Regularity Lemma.

An easy application of the Key Lemma shows that:

Observation 4.6. Suppose that there exists $\delta, 0 < \delta < 1$ such that for all $0 < \epsilon < 1$ and all $N \in \mathbb{N}$ there exist disjoint subsets $X_N, Y_N \subset P_1, |X_N| = |Y_N| \geq N$ such that (X_N, Y_N) is ϵ -regular with density δ . Then P_1 contains an infinite empty pair.

Proof. Apply the Key Lemma to each dual graph $(X_N, Y_N)'$, which is still regular and whose density remains bounded away from 0 and 1. For each $t < \omega$, for all N sufficiently large, $(X_N, Y_N)'$ contains a complete bipartite graph on t vertices, as this occurs as a subgraph of $R(t)$. \square

Lemma 4.7. Suppose that P_1 does not contain an infinite empty pair.

- (1) There is a function $f : (0, 1) \times \omega \rightarrow (0, 1)$ monotonic which approaches 1 as $\epsilon \rightarrow 0$ and $N \rightarrow \infty$ such that if (X, Y) is an ϵ -regular pair with $|X| = |Y| = n$ then $d(X, Y) \geq f(\epsilon, N)$.
- (2) There is a function $g : ((0, 1) \times \omega) \times \omega \rightarrow (0, 1)$, which is defined on all $((\epsilon, \ell), n)$ for which $n \geq N_{\epsilon, \ell}$, and which is monotonically increasing and approaches 1 as (ϵ, ℓ) stays fixed and $n \rightarrow \infty$, such that if $|X| = n$ then the density between any two regular components in an $(\epsilon, \ell)^*$ -decomposition of X is at least $g((\epsilon, \ell), n)$.
- (3) For every constant $c > 0$, and for all $\epsilon_0 > 0$, there exist $0 < \epsilon < \epsilon_0$ and for each such ϵ , cofinally many $\ell < \omega$ such that: for all n sufficiently large and all graphs X with

$|X| = n$, the number of interstitial edges in any $(\epsilon, \ell)^*$ -decomposition of X is strictly less than cn^2 .

Proof. (1) This restates Observation 4.6.

(2) The regularity lemma provides a decomposition in which all components are approximately the same size (± 1) , so the density of each ϵ -regular pair will be at least $f(\epsilon, \frac{n}{b_{\epsilon, \ell}})$.

It remains to prove (3). For the moment, let ϵ, ℓ be arbitrary and suppose that $|X| > N_{\epsilon, \ell}$. Then $|X| = n$ admits an ϵ -regular decomposition into at k -many pieces, each of size approximately $m = \frac{n}{k}$, where $(\dagger) \ell \leq k \leq \ell' := b_{\epsilon, \ell}$.

Writing $\delta := g((\epsilon, \ell), \frac{n}{\ell'})$, the contribution of the interstitial edges is at most:

$$\epsilon k^2 m^2 + (1 - \epsilon)(k)^2 (1 - \delta) m^2$$

where the term on the left assumes the irregular pairs are empty, and the term on the right counts the expected number of interstitial edges missing from the regular pairs. By (\dagger) , this in turn is bounded by:

$$\begin{aligned} &\leq \epsilon (\ell')^2 m^2 + (1 - \epsilon)(\ell')^2 (1 - \delta) m^2 \\ &\leq \epsilon (\ell')^2 \left(\frac{n}{\ell}\right)^2 + (1 - \epsilon)(\ell')^2 (1 - \delta) \left(\frac{n}{\ell}\right)^2 \\ &\leq n^2 \left(\frac{\ell'}{\ell}\right)^2 \left(\epsilon + (1 - \epsilon)(1 - \delta)\right) \end{aligned}$$

Thus our claim will hold whenever $\epsilon + (1 - \epsilon)(1 - \delta) < c \frac{\ell}{\ell'}$. To obtain this, choose $\epsilon > 0$ as small as desired and ℓ as large as desired. Then δ is monotonically increasing and approaches 1 as a function of n by (2), as desired. \square

We are now prepared to prove:

Theorem 4.8. *When φ is simple unstable, if there does not exist an infinite empty pair $X, Y \subset P_1$, then $\alpha(n) < \mathcal{O}(n^2)$.*

Proof. Given a positive real constant $c_0 > 0$, choose c, k, t such that $0 < c < 1$, $k, t \in \mathbb{N}$ and $c_0 > 2c + \frac{1}{k^t}$. Fix a pair (ϵ, ℓ) such that $\ell > k$ and (ϵ, ℓ) is one of the cofinally many pairs described in Lemma 4.7(3) for the constant c . Now, for n sufficiently large and any $|X| = n$, each of the components in an $(\epsilon, \ell)^*$ -decomposition of X will have size $> N_{\epsilon, \ell}$ so will themselves admit an $(\epsilon, \ell)^*$ -decomposition with few interstitial edges. Repeating this argument to an arbitrary depth we can confirm that the bulk of the ostensibly missing edges must continually vanish inside the (eventually) relatively much smaller components at each successive decomposition.

More precisely, let $\ell' := b_{\epsilon, \ell}$ and suppose $n \gg (N_{\epsilon, \ell})(\ell')^t$. By repeated application of Lemma 4.7(3) to each successive decomposition, we obtain the following upper bound on $\alpha(n)$, where $\ell \leq k_i \leq \ell'$ for each $1 \leq i \leq t$. The rightmost term assumes that after $t - 1$

levels of decomposition we obtain components which are themselves empty graphs.

$$\begin{aligned}
\alpha(n) &< cn^2 + c \left(\frac{n}{k_1} \right)^2 k_1 + c \left(\frac{n}{(k_2)^2} \right)^2 (k_2)^2 + \\
&\quad \dots + c \left(\frac{n}{(k_{t-1})^{t-1}} \right)^2 (k_{t-1})^{t-1} + \left(\frac{n}{(k_t)^t} \right)^2 (k_t)^t \\
&< cn^2 \left(1 + \frac{1}{k_1} + \frac{1}{(k_2)^2} + \dots + \frac{1}{(k_{t-1})^{t-1}} \right) + \left(\frac{n^2}{(k_t)^t} \right) \\
&< cn^2 \left(1 + \frac{1}{\ell} + \frac{1}{\ell^2} + \dots + \frac{1}{\ell^{t-1}} \right) + \left(\frac{n^2}{\ell^t} \right) \\
&< n^2 \left(\frac{\ell c}{\ell - 1} + \frac{1}{\ell^t} \right) \\
&< \left(2c + \frac{1}{\ell^t} \right) n^2 < c_0 n^2
\end{aligned}$$

by summing the convergent series. We have shown that for any constant c_0 , for all n sufficiently large $\alpha(n) < c_0 n^2$, so we finish. \square

Proof. (of Proposition 3.9) This is now an immediate corollary of Theorem 4.8, $\frac{n^2}{4}$ being the number of edges omitted in an empty pair. \square

Remark 4.9. *Theorem 4.8, and thus Proposition 3.9, are more natural than might appear. On one hand, as Szemerédi regularity deals with density, it cannot (in this formulation) give precise information about edge counts below $\mathcal{O}(n^2)$. On the other, the random graph contains many infinite empty pairs, for instance $(\{(a, z) : z \in M, z \neq a\}, \{(y, a) : y \in M, y \neq a\})$ when $\varphi = xRy \wedge \neg xRz$. One could imagine a future use for such theorems in suggesting ways of decomposing the parameter spaces of simple formulas into parts whose structure resembles random graphs (with many overlapping empty pairs) and parts whose structure is more cohesive, indicated by $\alpha(n) < \mathcal{O}(n^2)$.*

5. ORDER AND GENERICITY

In this section we show that P_1 , after localization, admits arbitrarily large ϵ -regular pairs of some fixed density bounded away from 0 and 1 precisely when P_2 , after localization, is unstable. Compare Conclusion 2.7.

Observation 5.1. *Suppose that for some $0 < \delta < 1$ and for all ϵ, n with $0 < \epsilon < 1$, $n \in \mathbb{N}$ we have a bipartite R -graph (X, Y) , $|X| = |Y| \geq n$, such that (X, Y) is ϵ -regular with density d , where $|d - \delta| < \epsilon$. Then R has the order property.*

Proof. It suffices to show that for arbitrarily small ϵ_0 and arbitrarily large k_0 there is a Szemerédi-regular decomposition of X and of Y into k_0 pieces such that all but $k_0(\epsilon_0)^2$ of the pairs X_i, Y_i are ϵ_0 -regular with density near δ . This is because we can apply the Key Lemma (in light of Remark 4.3) to conclude that any pattern which appears in the reduced graph corresponding to these components, in particular some given fragment of the order

property, actually occurs in X, Y . The subtlety is to ensure that the densities of the regular pairs are all approximately the same.

Given ϵ_0, k , let k_0, N_0 be the number of components and threshold size, respectively, given by the regularity lemma. Choose ϵ so that $\frac{1}{k_0} > \epsilon$ and $n > N_0$. Let (X, Y) be the ϵ -regular pair of size at least n and density near δ , given by hypothesis.

By regularity, $n > N_0$ means that there is a decomposition $X = \cup_{i \leq k_0} X_i$, $Y = \cup_{i \leq k_0} Y_i$ into disjoint pieces of near equal size and that all but $\epsilon_0(k_0)^2$ of the pairs (X_i, Y_j) are ϵ_0 -regular. However any one of these regular pairs (X_i, Y_j) will satisfy $|X_i|, |Y_j| = n/k_0 > \epsilon n$, so $|d(X_i, Y_j) - d(X, Y)| = |d(X_i, Y_j) - \delta \pm \epsilon| < \epsilon$ and $|d(X_i, Y_j) - \delta| < 2\epsilon$, as desired. \square

Recall that an equivalent definition of *SOP* is that there exists an indiscernible sequence $\langle a_i : i < \omega \rangle$ on which $\exists x (\neg \varphi(x; a_j) \wedge \varphi(x; a_i)) \iff j < i$. The main step in Shelah's classic proof that any unstable theory which does not have the independence property must have the strict order property can be characterized as follows:

Theorem E. (Shelah) *Let c be a finite set of parameters and $\langle a_i : i < \omega \rangle$ a c -indiscernible sequence. For $n < \omega$, any formula $\theta(x; \bar{z})$ and relations $R(x; y), R_1, \dots, R_n$ where $\ell(y) = \ell(a_i)$ and $R_i \in \{R(x; y), \neg R(x; y)\}$ for $i \leq n$, if*

$$i_1 < \dots < i_n \implies \exists x \left(\theta(x; c) \wedge R_1(x; a_{i_1}) \wedge \dots \wedge R_n(x; a_{i_n}) \right)$$

then either

- $\exists x \left(\theta(x; c) \wedge R_1(x; a_{i_{\sigma(1)}}) \wedge \dots \wedge R_n(x; a_{i_{\sigma(n)}}) \right)$ for any permutation $\sigma : n \rightarrow n$
- *some formula of T has the strict order property.*

The idea is to express the permutation σ as a sequence of swaps of successive elements (in the sense of the order $<$), and use the first instance, if any, where the swap produces inconsistency to obtain a sequence witnessing strict order. For details, see [7], Theorem II.4.7, pps. 70–72.

The subtlety in the corollary is to obtain not just the independence property but a bipartite random graph.

Corollary 5.2. *Suppose that $R(x; y)$ has the order property. If T does not have the strict order property, then there exist infinite disjoint sets A, B on which R is a bipartite random graph (i.e., $R(x; y)$ is I_2^2 in the sense of Definition 7.2 below).*

Proof. We first fix a template. Let M be a countable model of the theory of a bipartite random graph with two sorts P, Q and a single binary edge relation $E(x; y)$ with $E(x; y) \implies P(x) \wedge Q(y)$. Let $\langle x_i : i < \omega \rangle, \langle y_i : i < \omega \rangle$ be an enumeration of P and Q , respectively.

Now let $\langle a_i b_i : i < \omega \rangle$ be an indiscernible sequence on which R has the order property, i.e. $R(a_i, b_j) \iff i < j$. Suppose that for every $i < \omega$ we could find an element c_i such that for all $j < \omega$, $R(c_i, b_j) \iff E(x_i, y_j)$ in the template. Then setting $C := \bigcup_{i < \omega} c_i$, $B := \bigcup_{j < \omega} b_j$, (C, B) is a bipartite random graph for R .

So it remains to show that any finite subset p of the type $p_i(x) \in S(B)$ of any such c_i is consistent. Let η, ν be disjoint finite subsets of ω , and let $p(x) = \bigwedge_{j \in \eta} R(x; b_j) \wedge \bigwedge_{k \in \nu} \neg R(x; b_k)$. We are now in a position to apply Theorem E; as T is *NSOP*, $p(x)$ must be consistent. \square

Definition 5.3. Fix a binary edge relation R . Call a density $0 \leq \delta \leq 1$ attainable if for all ϵ there exists a sequence $\langle S_\epsilon^\delta = \langle (X_i, Y_i) : i < \omega \rangle$ of finite bipartite R -graphs such that for all $n < \omega, \epsilon > 0$ there is $N < \omega$ such that for all $i > N$,

- $|X_i| = |Y_i| \geq n$,
- (X_i, Y_i) is ϵ -regular with density d_i , where $|d_i - \delta| < \epsilon$.

Conclusion 5.4. Assume T does not have the strict order property. Then the following are equivalent for a binary relation $R(x, y)$:

- (1) For some $0 < \delta < 1$ and for all N, ϵ there exist disjoint X, Y with $|X| = |Y| \geq N$ such that (X, Y) is ϵ -regular with density d , $|d - \delta| < \epsilon$.
- (2) For any attainable $0 < \delta < 1$ such that for all N, ϵ there exist disjoint X, Y with $|X| = |Y| \geq N$ such that (X, Y) is ϵ -regular with density d , $|d - \delta| < \epsilon$.
- (3) R has the order property.

Proof. (2) \rightarrow (1) Attainable densities exist, e.g. $\frac{1}{2}$: consider subgraphs of an infinite random bipartite graph.

(1) \rightarrow (3) Observation 5.1.

(3) \rightarrow (1) Corollary 5.2, which says that from (3), assuming $NSOP$, we can construct an infinite random bipartite graph with edge relation R . \square

In other words, regularity plus compactness implies that density bounded away from 0, 1 gives any bipartite configuration including the order property, and model theory implies that the order property is enough to reverse the argument.

Corollary 5.5. Assume T does not have the strict order property, and $(T, \varphi) \mapsto \langle P_n \rangle$. Then the following are equivalent:

- (1) After localization, P_2 does not have the order property.
- (2) After localization, the density of any sufficiently large P_2 -regular pair (X, Y) must approach either 0 or 1. More precisely, there exists $f : \mathbb{N} \times (0, 1) \rightarrow [0, \frac{1}{2}]$ monotonic increasing as $n \rightarrow \infty, \epsilon \rightarrow 0$ such that if $X, Y \subset P_1$, $|X|, |Y| \geq n$ and (X, Y) is ϵ -regular, then either $d(X, Y) < f(n, \epsilon)$ or $d(X, Y) > 1 - f(n, \epsilon)$.

Corollary 5.6. If T is simple, then any characteristic sequence associated to one of its formulas satisfies the equivalent conditions of Corollary 5.5.

Proof. Conclusion 2.7. \square

Remark 5.7. The class of theories satisfying the equivalent conditions of Corollary 5.5 strictly contains the simple theories. Example 3.6 of [6] gives a formula with the tree property whose P_2 does not have the order property. This is essentially T_{feq}^* from [9]; basic examples of TP_2 will work.

Remark 5.8. Any formula with SOP_2 , also called TP_1 , has the order property in P_2 . For SOP_2 , see [9]. However, the next section suggests that more precise order properties may be useful.

6. TWO KINDS OF ORDER PROPERTY

Towards understanding the role of instability in the characteristic sequence, this section considers two polar opposite order properties and their implications for P_2 .

Definition 6.1. (Two kinds of order property) Let $\langle P_n \rangle$ be the characteristic sequence of φ .

- (1) φ has the n -compatible order property, for some $n < \omega$ (or $n = \infty$) if there exist $\langle a_i, b_i : i < \omega \rangle$ such that for all $m \leq n$ (or $m < \omega$), $P_{2m}(a_{i_1}, b_{j_1}, \dots, a_{i_m}, b_{j_m})$ iff $\max\{i_1, \dots, i_m\} < \min\{j_1, \dots, j_m\}$.
- (1)' When the sequence has support 2 this becomes: there exist $\langle a_i, b_i : i < \omega \rangle$ such that $P_2(a_i, a_j), P_2(b_i, b_j)$ for all i, j and $P_2(a_i, b_j)$ iff $i < j$.
- (2) φ has the n -empty order property, for some $n \in \omega$, if: there exist $\langle a_i, b_i : i < \omega \rangle$ such that (i) $P_2(a_i, b_j)$ iff $i < j$ and (ii) $\neg P_n(a_{i_1}, \dots, a_{i_n}), \neg P_n(b_{i_1}, \dots, b_{i_n})$ hold for all $i_1, \dots, i_n < \omega$.

Let us briefly justify not focusing on a natural third possibility, the “semi-compatible order property,” in which the elements $\langle a_i : i < \omega \rangle$ are an empty graph and the elements $\langle b_i : i < \omega \rangle$ are a positive base set.

Observation 6.2. There is a formula in the random graph which has the semi-compatible order property.

Proof. Choose two distinguished elements 0, 1 (this can be coded without parameters). Define $\psi(x; y, z)$ to be $x = y$ if $z = 0$, xRy otherwise. Then on any sequence of distinct elements $\langle a_i b_i : i < \omega \rangle \subset M$ which witness the order property ($a_i R b_j \iff i < j$), we have additionally that

$$\exists x (\psi(x; a_i, 0) \wedge \psi(x; b_j, 1)) \iff \exists x (x = a_i \wedge xRb_j) \iff i < j$$

so P_2 has the order property on the sequence $\langle (a_i, 0), (b_i, 1) : i < \omega \rangle$. On the other hand, $\exists x (x = a_i \wedge x = a_j) \iff i = j$, so the row of elements $(a_i, 0)$ is a P_2 -empty graph. Finally, $\exists x (xRb_i \wedge xRb_j)$ always holds, by the axioms of the random graph; so the row of elements $(b_j, 1)$ is a P_∞ -complete graph. \square

Observation 6.3. There is a formula in a simple rank 3 theory which has the 2-empty order property.

Proof. Let T be the theory of two crosscutting equivalence relations, E and F , each with infinitely many infinite classes and such that each intersection $\{x : E(a, x) \wedge F(x, b)\}$ is infinite. Let P be a unary predicate such that

- $(\forall x, y)(E(x, y) \wedge F(x, y) \implies P(x) \iff P(y))$
- For all $n < \omega$ and $y_1, \dots, y_k, y_{k+1}, \dots, y_n$ elements of distinct E -equivalence classes, there exists z such that $i \leq k \implies (\forall x)(E(x, y_i) \wedge F(x, z) \implies P(x))$ and $k < i \leq n \implies (\forall x)(E(x, y_i) \wedge F(x, z) \implies \neg P(x))$

Let $\psi(x; y, z)$ be $E(x, y) \wedge P(y)$ if $z = 0$, and $F(x, y) \wedge P(y)$ otherwise. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of elements chosen so that $(\forall x)(E(x, a_i) \wedge F(x, b_j) \implies P(x))$ iff $i < j$. Then it is easy to see ψ has the 2-empty order property on the sequence $\langle (a_i, 0), (b_i, 1) : i < \omega \rangle$. \square

Remark 6.4. Assuming $MA + 2^{\aleph_0} > \aleph_1$, Shelah has constructed an ultrafilter on ω which saturates (small) models of the random graph, but not of theories with the tree property ([7] Theorem VI.3.9). This is a strong argument for the “semi-compatible order property” being less complex: it cannot, by itself, imply maximality in the Keisler order, whereas we will see that the ∞ -compatible order property does. It may still be that persistence, in the sense of [6], of any order property in P_2 creates complexity.

We return to the study of the compatible order property.

Convention 6.5. When more than one characteristic sequence is being discussed, write $P_n(\varphi)$ to indicate the n th hypergraph associated to the formula φ . Recall that φ_ℓ is shorthand for $\bigwedge_{1 \leq i \leq \ell} \varphi(x; y_i)$.

The following general principle will be useful.

Claim 6.6. Suppose that we have a sequence $C := \langle c_i : i \in \mathbb{Z} \rangle$ and a formula $\rho(x; y, z)$ such that:

- (1) $\exists x \rho(x; c_i, c_j) \iff i < j$
- (2) $\exists x (\bigwedge_{\ell \leq n} \rho(x; c_{i_\ell}, c_{j_\ell}))$ just in case $\max\{i_1, \dots, i_n\} < \min\{j_1, \dots, j_n\}$

Then ρ has the ∞ -compatible order property.

Proof. By compactness, it is enough to show that there are elements $\langle \alpha_i, \beta_i : i < n \rangle$ witnessing a fragment of the ∞ -compatible order property of size n .

Define $\alpha_1 \dots \alpha_n, \beta_1, \dots, \beta_n$ as follows. Remark 6.7 provides a picture.

- $\alpha_i := c_{2i-1}c_{4n-2i+1}$, $1 \leq i \leq n$
- $\beta_i := c_{-2i}c_{2i}$, $1 \leq i \leq n$

Then $P_1(\alpha_i), P_1(\beta_i)$ for $1 \leq i \leq n$ by (1). For all $1 \leq k, r \leq n$ with $r + k = m$, condition (2) says that $P_m(\alpha_{i_1}, \dots, \alpha_{i_k}, \beta_{j_1}, \dots, \beta_{j_r})$ iff

$$\max\{2\ell : \ell \in \{j_1, \dots, j_r\}\} < \min\{2s - 1 : s \in \{i_1, \dots, i_k\}\}$$

that is, iff $\max\{j_1, \dots, j_r\} < \min\{i_1, \dots, i_k\}$, so we are done. \square

Remark 6.7. The ∞ -compatible order property describes an interaction between two P_∞ -complete graphs, i.e. consistent types. The hypotheses (1)-(2) of Claim 6.6 are enough to allow a weak description of intervals. That is, we choose the sequences α_i, β_i to each describe a concentric sequence of intervals (each α_i, β_i corresponds to a set of matching parentheses) along the sequence $\langle c_i \rangle$:

$$\leftarrow [-[-[-[-[-]]-]]-] - \dots - -(-(-(-(-)-)-)-) \rightarrow$$

which we can interlace to obtain ∞ -c.o.p. by judicious choice of indexing:

$$\leftarrow [-[-[-[-(-)-(-)-(-)-(-)-)]-]] \rightarrow$$

In this picture, the enumeration of the α s (\leftarrow), would proceed from the outmost pair to the inmost and the enumeration of the β s (\rightarrow) from inmost to outmost.

Observation 6.8. Suppose that φ has the strict order property, i.e. there is an infinite sequence $\langle c_i : i < \omega \rangle$ on which $\exists x (\neg \varphi(x; c_i) \wedge \varphi(x; c_j)) \iff i < j$. Then $\neg \varphi(x; y) \wedge \varphi(x; z)$ has the ∞ -compatible order property.

Proof. Writing $\rho(x; y, z) = \neg \varphi(x; y) \wedge \varphi(x; z)$,

- $\exists x \rho(x; c_i, c_j) \iff i < j$, by definition of strict order;
- $\exists x (\rho(x; c_i, c_j) \wedge \rho(x; c_k, c_\ell)) \iff i, k < j, \ell$

and the characteristic sequence $P_\infty(\rho)$ has support 2. Apply Claim 6.6. \square

Example 6.9. The theory T of the triangle-free random graph with edge relation R has the ∞ -c.o.p. Consider $\varphi(x; y, z) = xRy \wedge xRz$. (The negative instances could be added but are not necessary.) Then:

- $P_1((y, z)) \iff \neg yRz$.
- $P_2((y, z), (y', z')) \iff \{y, y', z, z'\}$ is an empty graph.
- The sequence has support 2, as the only problems come from a single new edge:
 $P_n((y_1, z_1), \dots, (y_n, z_n)) \iff$

$$\exists x \left(\bigwedge_{i \leq n} xRy_i \wedge \bigwedge_{j \leq n} xRz_j \right) \text{ that is, if } \bigcup_i y_i \cup \bigcup_j z_j \text{ is a } P_2\text{-empty graph.}$$

Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence witnessing the 2-empty order property with respect to the edge relation R , say $a_i R b_j \iff j \leq i$. Then $\exists x(xRa_i \wedge xRb_j) \iff i < j$, i.e. $(a_i, b_j) \in P_1 \iff i < j$. Also, $\exists x(xRa_i \wedge xRb_j \wedge xRa_k \wedge xRb_\ell) \iff$ if, in addition, $i, k < j, \ell$. Apply Claim 6.6.

Finally, we tie the compatible order property to SOP_3 , a model-theoretic rigidity property. SOP_3 will be discussed extensively in the next section, Definition 7.4, Definition 7.5.

Lemma 6.10. Suppose that $\theta(x; y)$ has SOP_3 in the sense of Definition 7.4, so $\ell(x) = \ell(y)$. Let $\varphi_r = \varphi$, $\psi_\ell = \psi$ be the formulas from Definition 7.5. Then $\rho(x; y, z) := \varphi_r(y, x) \wedge \psi_\ell(x, z)$ has the ∞ -compatible order property on some $A' \subset P_1$. Moreover, we can choose A' so that the sequence restricted to A' has support 2.

Remark 6.11. This is an existential assertion, and it is straightforward to check that it remains true if we modify ρ to include the corresponding negative instances.

Proof. (of Lemma) Let $A := \langle a_i : i < \mathbb{Q} \rangle$ be an infinite indiscernible sequence from Definition 7.5. Then

$$P_1((a_i, a_j)) \iff \exists x (\varphi_r(a_i, x) \wedge \psi_\ell(x, a_j)) \iff i < j$$

by the choice of φ, ψ . More generally,

$$P_n((a_{i_1}, a_{j_1}), \dots, (a_{i_n}, a_{j_n})) \iff \exists x \left(\bigwedge_{t \leq n} \varphi_r(x; a_{i_t}) \wedge \bigwedge_{t \leq n} \psi_\ell(x; a_{j_t}) \right)$$

which, again applying Definition 7.5, happens iff $\max\{i_1, \dots, i_n\} < \min\{j_1, \dots, j_n\}$, a condition which has support 2. We now apply Claim 6.6 to obtain $A' \subset A \times A$ witnessing the compatible order property. Note that while $\langle P_n \rangle$ need not depend on 2 elsewhere in P_1 (we know very little about ρ off A), it does depend on 2 on elements from the sequence A' . \square

Observation 6.12. Suppose $\theta(x; y)$ has the ∞ -compatible order property. Then the formula $\varphi(x; y, z) := \theta(x; y) \wedge \neg\theta(x; z)$ has SOP_3 .

Proof. Let $\langle a_i b_i : i < \omega \rangle$ be a sequence witnessing the ∞ -compatible order property; this will play the role of the sequence $\langle \bar{a}_i : i < \omega \rangle$ from Definition 7.5. In the notation of that Definition, let $\varphi(x; y, z) := \theta(x; y) \wedge \neg\theta(x; z)$ and $\psi(x; y, z) := \theta(x; z)$. We check the conditions.

(1) Clearly $\{\varphi(x; y, z), \psi(x; y, z)\}$ is inconsistent.

(3) When $i > j$, $\{\varphi(x; a_i b_i), \psi(x; a_j b_j)\} = \{\theta(x; a_i) \wedge \neg\theta(x; b_i), \theta(x; b_j)\}$ is inconsistent because $\neg P_2(a_i, b_j)$.

Finally, for $1 \leq j < \omega$ let $p_j(x) = \{\theta(x; a_i) : 1 \leq i \leq j\} \cup \{\theta(x; b_\ell) : j < \ell < \omega\}$. The ∞ -c.o.p. implies $P_n(a_1, \dots, a_j, b_{j+1}, \dots, b_n)$ for all $n < \omega$, so p_j is consistent. However, $i < j \implies \neg P_2(b_i, a_j)$ so $p_{j,n}(x) \vdash \neg \theta(x; b_i)$ for each $1 \leq i \leq j$. Choosing $c_j \models p_j$ for each $j < \omega$ gives (2). \square

Remark 6.13. Applying Shelah's theorem that any theory with SOP_3 is maximal in the Keisler order [8], [9], we conclude that if T contains a formula φ with the ∞ -compatible order property, then T is maximal in the Keisler order. For more on Keisler's order, see [5].

7. CALIBRATING RANDOMNESS

In this final section, we observe and explain a discrepancy between the model-theoretic notion of an infinite random k -partite graph and the finitary version given by Szemerédi regularity, showing essentially that a class of infinitary k -partite random graphs which do not admit reasonable finite approximations must have the strong order property SOP_3 (a model-theoretic notion of rigidity, Definition 7.4 below).

7.1. A seeming paradox.

Observation 7.1. *Let T be the theory of the triangle-free random graph, with edge relation R . Then it is consistent with T that there exist disjoint infinite sets X, Y, Z such that each pair $(X, Y), (Y, Z), (X, Z)$ is a bipartite random graph.*

Proof. The construction has countably many stages. At stage 0, let $X_0 = \{a\}, Y_0 = \{b\}, Z_0 = \{c\}$ where a, b, c have no R -edges between them. At stage $i + 1$, let X_{i+1} be X_i along with $2^{|Y_i|+|Z_i|}$ -many new elements:

- (1) for each subset $\tau \subset Y_i$, a new element x_τ such that for $y \in Y$, $x_\tau R y \iff y \in \tau$, however $\neg x_\tau R x$ for any x previously added to X_{i+1} .
- (2) for each subset $\nu \subset Z_i$, a new element x_ν such that for $z \in Z$, $x_\nu R z \iff z \in \nu$, with x_ν likewise R -free from previous elements of X_{i+1} .

Y_{i+1}, Z_{i+1} are defined symmetrically. As we are working in the triangle-free random graph, in order that the construction be able to continue, it is enough that the sets X_i, Y_i, Z_i are each empty graphs, i.e., at no point do we ask for a triangle.

To finish, set $X = \bigcup_i X_i, Y = \bigcup_i Y_i, Z = \bigcup_i Z_i$. Each pair is a bipartite random graph, as desired. \square

But recall:

Theorem F. (weak version of Key Lemma, Theorem D) *Fix $1 > \delta > 0$ and a binary edge relation R . Then there exist $\epsilon' = \epsilon'(\delta), N' = N'(\epsilon', \delta)$ such that: if $\epsilon < \epsilon', N > N', X, Y, Z$ are disjoint finite sets of size at least N , and each of the pairs $(X, Y), (Y, Z), (X, Z)$ is ϵ -regular with density δ , then there exist $x \in X, y \in Y, z \in Z$ so that x, y, z is an R -triangle.*

Obviously, we cannot have an R -triangle in the triangle-free random graph. Nonetheless each of the pairs (X, Y) in Observation 7.1 manifestly has finite subgraphs of any attainable density.

The difficulty comes when we try to choose finite subgraphs $X' \subset X, Y' \subset Y, Z' \subset Z$ so that the densities of all three pairs are *simultaneously* near the same $\delta > 0$. If (X', Y') and (Y', Z') are reasonably dense, (X', Z') will be near 0. Put otherwise, we may choose elements of X independently over Y , and independently over Z , but not both at the same time.

The constructions below generalize this example, and give a way of measuring the “depth” of independence in a constellation of sets X_1, \dots, X_n , where any pair (X_i, X_j) is a bipartite random graph. The example of the triangle-free random graph is paradigmatic: we shall see that a bound on the depth of independence will produce the 3-strong order property SOP_3 .

7.2. Constellations of independence properties.

Definition 7.2. Fix a formula $R(x; y)$.

- (1) Let A, B be disjoint sets of k - and n -tuples respectively, where $k = \ell(x), n = \ell(y)$. Then A is independent over B with respect to R just in case for any two finite disjoint $\eta, \nu \subset B$, there exists $a \in A$ such that $b \in \eta \rightarrow R(a; b)$ and $b \in \nu \rightarrow \neg R(a; b)$.
- (2) Let A_1, \dots, A_k be disjoint sets (of m -tuples, where $m = \ell(x) = \ell(y)$). Then A_1 is independent over A_2, \dots, A_k with respect to R just in case A_1 is independent over $B := \bigcup_{2 \leq i \leq k} A_i$ in the sense of (2).
- (3) $R(x; y)$ is a bipartite random graph if there exist disjoint infinite sets A, B such that A and B are each independent over the other wrt R .
- (4) $R(x; y)$ is I_k^m , for some $2 \leq k \leq m$, if there exist disjoint infinite sets $\langle A_i : i < m \rangle$ such that for any distinct $i_1, \dots, i_k < \omega$, A_{i_1} is independent over $\bigcup_{2 \leq j \leq k} A_{i_j}$ w.r.t. R . Note that k refers to the depth of the independence, and not the size of the finite disjoint η, ν .

Observation 7.3. Let $R(x; y)$ be a symmetric formula. The following are equivalent.

- (1) R is I_ω^ω .
- (2) There is an infinite subset of the monster model on which R is a random graph. (Certainly this need not be definable or interpretable in any way).

Definition 7.4. (Shelah, [8]:Definition 2.5) For $n \geq 3$, the theory T has SOP_n if there is a formula $\varphi(x; y)$, $\ell(x) = \ell(y) = k$, $M \models T$ and a sequence $\langle a_i : i < \omega \rangle$ with each $a_i \in M^k$ such that:

- (1) $M \models \varphi(a_i, a_j)$ for $i < j < \omega$
- (2) $M \models \neg \exists x_1, \dots, x_n (\bigwedge \{\varphi(x_m, x_k) : m < k < n \text{ and } k = m + 1 \text{ mod } n\})$

Theorem G. (Shelah, [8]: (1) is Claim 2.6, (2) is Theorem 2.9)

- (1) For a theory T , $SOP \implies SOP_{n+1} \implies SOP_n$, for $n \geq 3$ (not necessarily for the same formula).
- (2) If T is a complete theory with SOP_3 , then T is maximal in the Keisler order.

We will derive SOP_3 from failures of randomness, using the following equivalent definition. Remember that, by convention, a_i, x, \dots need not be singletons.

Definition 7.5. ([9]:Fact 1.3) T has SOP_3 iff there is an indiscernible sequence $\langle a_i : i < \omega \rangle$ and \mathcal{L} -formulas $\varphi(x; y), \psi(x; y)$ such that:

- (1) $\{\varphi(x; y), \psi(x; y)\}$ is contradictory.
- (2) there exists a sequence of elements $\langle c_j : j < \omega \rangle$ such that
 - $i \leq j \implies \varphi(c_j; a_i)$
 - $i > j \implies \psi(c_j; a_i)$
- (3) if $i < j$, then $\{\varphi(x; a_j), \psi(x; a_i)\}$ is contradictory.

The idea of the construction (Theorem 7.7) is contained in the following straightforward example.

Example 7.6. *Let T be the triangle free random graph, with edge relation R . Then R is I_2^3 but not I_3^3 , and T is SOP_3 .*

Proof. Let us prove the final clause (for the rest see Observation 7.1 and the discussion following).

The theory by definition contains a forbidden configuration, a triangle. Suppose A, B, C are disjoint infinite sets witnessing I_2^3 . Let us construct a sequence of triples $S = \langle a_i, b_i, c_i : i < \omega \rangle$ such that, for $i < \omega$,

- For all $j \leq i$, $b_i R a_j$.
- For all $j \leq i$, $c_i R b_j$.
- For all $j \leq i$, $a_{i+1} R c_j$.

Define a binary relation $<_\ell$ on triples by:

$$(x, y, z) \leq_\ell (x', y', z') \iff ((x R y' \wedge y R z' \wedge z R x'))$$

While $<_\ell$ need not be a partial order on the model, it does linearly order the sequence S by construction. Looking towards Definition 7.5, let us define two new formulas (the variables t stand for triples):

- $\varphi(t_0; t_1, t_2) = t_1 <_\ell t_2 <_\ell t_0$
- $\psi(t_0; t_1, t_2) = t_0 <_\ell t_1 <_\ell t_2$

Let us check that these formulas give SOP_3 . For condition (1), $\varphi(t_0; t_1, t_2), \psi(t_0; t_1, t_2)$ means that $(x_0, y_0, z_0) <_\ell (x_1, y_1, z_1) <_\ell (x_2, y_2, z_2) <_\ell (x_0, y_0, z_0)$. Then $x_i R y_j, y_j R z_k, z_k R x_i$ which gives a triangle, contradiction.

It is straightforward to satisfy (2) by compactness (e.g. by choosing S codense in a larger indiscernible sequence).

Finally, for condition (3), suppose $i < j$ but $\varphi(t; \gamma_i), \psi(t; \gamma_j)$ is consistent, where $t = (x, y, z)$. This means that $(x, y, z) <_\ell (a_i, b_i, c_i) <_\ell (a_j, b_j, c_j) <_\ell (x, y, z)$ (where the middle $<_\ell$ comes from the behavior of $<_\ell$ on the sequence S). As in condition (1), this gives a triangle, contradiction. \square

We can extend this idea to a much larger engine for producing the rigidity of SOP_3 from a forbidden configuration.

Theorem 7.7. *Suppose that for some $2 \leq n < \omega$, the formula R of T is I_n^{n+1} but not I_{n+1}^{n+1} . Then T is SOP_3 .*

Proof. The construction is arranged into four stages.

Step 1: Finding a universally forbidden configuration G .

By hypothesis, R is not I_{n+1}^{n+1} . This means that the infinitary type $p(X_0, \dots, X_n)$, which describes $n+1$ infinite sets X_i which are I_{n+1}^{n+1} in the sense of Definition 7.2, is not consistent. Let G be a finite inconsistent subset of height h , in the variables $V_G = \{x_j^i : 1 \leq i \leq h, 0 \leq j \leq n\}$, and described by the edge map $E_G : \{((i, j), (i', j')) : i, i' \leq h, j \neq j' \leq n\} \rightarrow \{0, 1\}$. As the inconsistency of p is a consequence of T , G will be a universally forbidden configuration:

$$(1) \quad T \vdash \neg(\exists x_0^1, \dots, x_n^h) \left(\bigwedge_{i, i' \leq h, j \neq j' \leq n} R(x_j^i, x_{j'}^{i'}) \iff E((i, j), (i', j')) = 1 \right)$$

Note that the configuration remains agnostic on edges between elements in the same column, in keeping with the definition of I_ℓ^m .

In what follows G will appear as a template which we shall try to approximate using I_n^{n+1} . Here are the vertices of G arranged as they will be visually referenced (the edges are not drawn in):

$$\begin{array}{ccccc} x_0^h & & x_k^h & & x_n^h \\ \vdots & & \vdots & & \vdots \\ x_0^\rho & & x_k^\rho & & x_n^\rho \\ \vdots & & \vdots & & \vdots \\ x_0^1 & \dots & x_k^1 & \dots & x_n^1 \end{array}$$

FIGURE 1. Vertices of the forbidden configuration G , arranged in columns. When comparing this configuration to an array whose rows are indexed modulo h , the superscript of the top column becomes 0.

Step 2: Building an array A of approximations to G .

Let A_0, \dots, A_n be disjoint infinite sets witnessing I_n^{n+1} for R . As in Example 7.6, we will use elements from these columns A_i to build an array $A = \langle a_i^\rho : 1 \leq \rho < \omega, 0 \leq i \leq n \rangle$. Fixing notation,

- $a_0^\rho, \dots, a_n^\rho$ is called a *row*.
- $\text{Col}(i) = \{j : j \neq i, i+1 \pmod{n+1}\}$ is the set of column indices associated to the column index i .
- Define an ordering on pairs of indices (β for “before”):

$$\beta((t', i'), (t, i)) \iff_{\text{def}} \left((t' < t \wedge i' \in \text{Col}(i)) \vee (t' = t \wedge i' < i) \right)$$

Claim 7.8. *We may build the array A to satisfy:*

- (1) For all ρ , $a_k^\rho \in A_k$.
- (2) For any ρ', ρ, k, k' such that $\beta((\rho', k'), (\rho, k))$,

$$a_k^\rho R a_{k'}^{\rho'} \iff E_G((r, k), (r', k')) = 1$$

where $r \equiv \rho \pmod{h}$, $r' \equiv \rho' \pmod{h}$.

Proof. We choose elements in a helix $(a_0^1, a_1^1, \dots, a_n^1, a_0^2, a_1^2, \dots)$ so that $\beta((\rho', k'), (\rho, k))$ implies that $a_{k'}^{\rho'}$ is chosen before a_k^ρ .

When the time comes to choose a_k^ρ , we look for an element of A_k which satisfies Condition (2) of the Claim, that is, which, by Condition (1), realizes a given R -type over disjoint

finite subsets of the columns A_i ($i \in \text{Col}(k)$). As (A_0, \dots, A_n) was chosen to be I_n^{n+1} and $|\text{Col } k| = n - 1$, an appropriate a_k^ρ exists. \square

Step 3: Defining the relation $<_\ell$, which has no pseudo- $(n+1)$ -loops.

We now define a binary relation $<_\ell$ on m -tuples, where $m = h(n+1)$. Fix the enumeration of these tuples to agree with the natural interpretation as blocks B_ℓ of h consecutive rows in the array A (see Figure 7.2). That is, write the variables $Y := \langle y_i^t : 1 \leq t \leq h, 0 \leq i \leq n \rangle$, $Z := \langle z_{i'}^{t'} : 1 \leq t' \leq h, 0 \leq i' \leq n \rangle$. Define:

$$\mathbf{Y} <_\ell \mathbf{Z} \iff_{(def)} \bigwedge_{1 \leq t', t \leq h, 0 \leq i, i' \leq n} (i' \in \text{Col}(i)) \implies \left(z_i^t R y_{i'}^{t'} \iff E_G((t, i), (t', i')) = 1 \right)$$

Let B be a partition of the array A into blocks B_k ($k < \omega$) each consisting of h consecutive rows, so $B_k := \langle a_t^r : 0 \leq t \leq n, kh + 1 \leq r \leq (kh) + h \rangle$, for each $k < \omega$ (see Figure 7.2). By Claim 7.8, $i \prec j \implies B_i <_\ell B_j$.

Definition 7.9. A pseudo- $(n+1)$ -loop is a sequence W_i ($0 \leq i \leq n$) such that for some m , $1 \leq m < n$:

$$(2) \quad \left(\bigwedge_{(0 < j < i \leq n)} W_j <_\ell W_i \right) \wedge \left(\bigwedge_{1 \leq j \leq m} W_0 <_\ell W_j \right) \wedge \left(\bigwedge_{m < j \leq n} W_j <_\ell W_0 \right)$$

Suppose it were consistent with T to have blocks of variables $W_0 \dots W_n$ which form a pseudo- $(n+1)$ -loop. Write $W_k(i) = \{w_i^{hk+1}, \dots, w_i^{hk+h}\}$ for the i th column of block W_k . Figure 7.2 gives the picture, where the elements a are replaced by variables w and the blocks B_i become W_i . Set $W_G = W_0(0) \cup \dots \cup W_n(n)$ (which can be visualized as the boldface columns in Figure 7.2).

By definition of $<_\ell$, the pseudo- $(n+1)$ -loop (2) implies that whenever

$$((j \in \text{Col}(i)) \wedge ((0 < j < i \leq n) \vee (j = 0 \wedge i \leq m) \vee (m < j \wedge i = 0)))$$

we will have:

$$\left(\forall w_k^t \in W(i), w_{k'}^{t'} \in W(j) \right) \left(w_k^t R w_{k'}^{t'} \iff E_G((t, k), (t', k')) = 1 \right)$$

In other words, $<_\ell$ says that on certain pairs of elements in our proposed instance W_G of G , namely those elements whose respective columns “fall into each other’s scope” as given by the Col operator, W_G faithfully follows the template of G . It is easy to check that in a pseudo- $(n+1)$ -loop every pair $j \neq i$ in $\{0, \dots, n\}$ has this property. Thus pseudo- $(n+1)$ -loops in $<_\ell$ are inconsistent with T .

Step 4: Obtaining SOP_3 .

Step 3 showed that our array A of approximations had a certain rigidity, which we can now identify as SOP_3 . Following Definition 7.5, let us define $\varphi_r(x; y_1, \dots, y_n)$ and $\psi_\ell(x; y_1, \dots, y_n)$,

$$B_{\ell_n} =$$

$$B_{\ell_k} =$$

$$B_1 =$$

$$B_0 =$$

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where the the variables are blocks, and the subscripts “ ℓ ” and “ r ” are visual aids: the element x goes to the left of the elements y_i under ψ , and to their right under φ .

That is, we set:

$$\begin{aligned} \bullet \varphi_r(x; y_1, \dots y_n) &= \bigwedge_{1 \leq i \neq j \leq n} y_i <_\ell y_j \ \wedge \ \bigwedge_{1 \leq i \leq n} y_i <_\ell x \\ \bullet \psi_\ell(x; y_1, \dots y_n) &= \bigwedge_{1 \leq i \leq n} x <_\ell y_i \ \wedge \ \bigwedge_{1 \leq i \neq j \leq n} y_i <_\ell y_j \end{aligned}$$

Now let us verify that the conditions of Definition 7.5 hold. Let B be the sequence of blocks defined in Step 3, and assume without loss of generality that $B = \langle B_k : k < \omega \rangle$ is indiscernible and moreover is dense and codense in some indiscernible sequence B' . Let $A = \langle A_i : i < \omega \rangle$ be an indiscernible sequence of n -tuples of elements of B .

- (1) $\{\varphi_r(x; y_1, \dots y_n), \psi_\ell(x; y_1, \dots y_n)\}$ is contradictory because it gives rise to a pseudo- $(n+1)$ -loop.
- (2) By construction, for any $k < \omega$, the type

$$\{\psi_\ell(x; A_j) : j \leq k\} \cup \{\varphi_r(x; A_i) : k < i\}$$

is consistent, because $<_\ell$ linearly orders B , thus also B' . Choose the desired sequence of witnesses to be elements in the indiscernible sequence B' which are interleaved with B .

- (3) Suppose we have $\{\varphi_r(x; A_j), \psi_\ell(x; A_i)\}$ for some $i < j$, or in other words:

$$\{\varphi_r(x; B_{j_1}, \dots B_{j_n}), \psi_\ell(x; B_{i_1}, \dots B_{i_n})\} \text{ where } \{i_1, \dots i_n\} < \{j_1, \dots j_n\}$$

Then $x <_\ell B_{i_1} <_\ell \dots <_\ell B_{i_n} <_\ell B_{j_1} <_\ell \dots <_\ell B_{j_n} <_\ell x$ is a pseudo- $(2n+1)$ -loop (remember that $<_\ell$ holds between any increasing pair of elements of B by construction). Thus a fortiori we have a pseudo- $(n+1)$ -loop, contradicting the conclusion of Step 3.

We have shown that the theory T has SOP_3 , so we finish. \square

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